

# ON $M$ – TERMS APPROXIMATIONS BESOV CLASSES IN LORENTZ SPACES

G. AKISHEV

**Abstract.** In this paper we consider Lorentz space with a mixed norm of periodic functions of many variables. We obtain the exact estimation of the best  $M$ -term approximations of Nikol'ski's, Besov's classes in the Lorentz space with the mixed norm.

**Keywords:** Lorentz space, and Besov's class, and approximation

**MSC:** 41A10 and 41A25

## 1. INTRODUCTION

Let  $\bar{x} = (x_1, \dots, x_m) \in \mathbb{T}^m = [0, 2\pi)^m$  and  $\theta_j, p_j \in [1, +\infty)$ ,  $j = 1, \dots, m$ . Let  $L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m)$  denotes the space of Lebesgue measurable functions  $f(\bar{x})$  defined on  $\mathbb{R}^m$ , which have  $2\pi$  – period with respect to each variable such that

$$\|f\|_{\bar{p}, \bar{\theta}} = \|\dots\|f\|_{p_1, \theta_1} \dots \|f\|_{p_m, \theta_m} < +\infty,$$

where

$$\|g\|_{p, \theta} = \left\{ \int_0^{2\pi} (g^*(t))^{\theta} t^{\frac{\theta}{p}-1} dt \right\}^{\frac{1}{\theta}},$$

where  $g^*$  a non-increasing rearrangement of the function  $|g|$  (see. [1]).

It is known that if  $\theta_j = p_j$ ,  $j = 1, \dots, m$ , then  $L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m) = L_{\bar{p}}(\mathbb{T}^m)$  the Lebesgue measurable space of functions  $f(\bar{x})$  defined on  $\mathbb{R}^m$ , which have  $2\pi$  – period with respect to each variable with the norm

$$\|f\|_{\bar{p}} = \left[ \int_0^{2\pi} \left[ \dots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} dx_m \right]^{\frac{1}{p_m}} < +\infty,$$

where  $\bar{p} = (p_1, \dots, p_m)$ ,  $1 \leq p_j < +\infty$ ,  $j = 1, \dots, m$  (see [2], p. 128).

Any function  $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$  can be expanded to the Fourier series

$$\sum_{\bar{n} \in \mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where  $a_{\bar{n}}(f)$  Fourier coefficients of  $f \in L_1(\mathbb{T}^m)$  with respect to multiple trigonometric system  $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\bar{n} \in \mathbb{Z}^m}$ , and  $\mathbb{Z}^m$  is the space of points in  $\mathbb{R}^m$  with integer coordinates.

For a function  $f \in L(\mathbb{T}^m)$  and a number  $s \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$  let us introduce the notation

$$\delta_0(f, \bar{x}) = a_0(f), \quad \delta_s(f, \bar{x}) = \sum_{\bar{n} \in \rho(s)} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where  $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$ ,

$$\rho(s) = \left\{ \bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : [2^{s-1}] \leq \max_{j=1, \dots, m} |k_j| < 2^s \right\},$$

where  $[a]$  is the integer part of the number  $a$ .

Let us consider Nikol'skii, Besov classes (see [2], [3]). Let  $1 < p_j < +\infty, 1 < \theta_j < +\infty, j = 1, \dots, m, 1 \leq \tau \leq \infty$ , and  $r > 0$

$$H_{\bar{p}, \bar{\theta}}^r = \left\{ f \in L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m) : \sup_{s \in \mathbb{Z}_+} 2^{sr} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}} \leq 1 \right\},$$

$$B_{\bar{p}, \bar{\theta}, \tau}^r = \left\{ f \in L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m) : \left( \sum_{s \in \mathbb{Z}_+} 2^{sr\tau} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}}^\tau \right)^{\frac{1}{\tau}} \leq 1 \right\}.$$

It is known that for  $1 \leq \tau \leq \infty$  the following holds

$$B_{\bar{p}, \bar{\theta}, 1}^r \subset B_{\bar{p}, \bar{\theta}, \tau}^r \subset B_{\bar{p}, \bar{\theta}, \infty}^r = H_{\bar{p}, \bar{\theta}}^r.$$

Let  $f \in L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m)$  and  $\{\bar{k}^{(j)}\}_{j=1}^M$  be a system of vectors  $\bar{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$  with integer coordinates. Consider the quantity

$$e_M(f)_{\bar{p}, \bar{\theta}} = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^M b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}},$$

where  $b_j$  are arbitrary numbers. The quantity  $e_M(f)_{\bar{p}, \bar{\theta}}$  is called the best  $M$  – term approximation of a function  $f \in L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m)$ . For a given class  $F \subset L_{\bar{p}, \bar{\theta}}(\mathbb{T}^m)$  let

$$e_M(F)_{\bar{p}, \bar{\theta}} = \sup_{f \in F} e_M(f)_{\bar{p}, \bar{\theta}}.$$

The best  $M$  – term approximation was defined by S.B. Stechkin [4]. Estimations of  $M$  – term approximations of different classes were provided by R.S. Ismagilov [5], E.S. Belinsky [6], V.E. Maiorov [7], B.S. Kashin [8], R. DeVore [9], V.N. Temlyakov [10], A.S. Romanyuk [11], Dinh Dung [12], Wang Heping and Sun Yongsheng [13], L. Q. Duan and G.S. Fang [14], W. Sickel and M. Hansen [15], S.A. Stasyuk [16], [17] and others (see bibliography in [18], [19], [20]).

For the case  $p_1 = \dots = p_m = p$  and  $q_1 = \dots = q_m = \theta_1 = \dots = \theta_m = q$  R.A. DeVore and V.N. Temlyakov [20] proved the following theorem.

**Theorem A.** *Let  $1 \leq p, q, \tau \leq \infty$  and  $r(p, q) = m \left( \frac{1}{p} - \frac{1}{q} \right)_+$  if  $1 \leq p \leq q \leq 2$ , or  $1 \leq q \leq p < \infty$  and  $r(p, q) = \max \left\{ \frac{m}{p}, \frac{m}{2} \right\}$  in other cases. Then for  $r > r(p, q)$  the following holds*

$$e_M(B_{p, \tau}^r)_q \asymp M^{-\frac{r}{m} + \left( \frac{1}{p} - \max \left\{ \frac{1}{q}, \frac{1}{2} \right\} \right)_+},$$

where  $a_+ = \max \{a; 0\}$ .

Moreover, in the case of  $m \left( \frac{1}{p} - \frac{1}{q} \right) < r < \frac{m}{p}, 1 < p \leq 2 < q < \infty$  S.A. Stasyuk [16] proved  $e_M(B_{p, \tau})_q \asymp M^{-\frac{q}{2} \left( \frac{r}{m} - \left( \frac{1}{p} - \frac{1}{q} \right) \right)}$ . In the case  $r = \frac{m}{p}$  obtained  $e_M(B_{p, \tau})_q \asymp M^{-\frac{1}{2}} (\log M)^{1 - \frac{1}{\tau}}$  (see [17]).

The main goal of the present paper is to find the order of the quantity  $e_M(F)_{\bar{q}, \bar{\theta}}$  for the class  $F = B_{\bar{p}, \bar{\theta}, \tau}^r$ .

The notation  $A(y) \asymp B(y)$  means that there exist positive constants  $C_1, C_2$  such that  $C_1 A(y) \leq B(y) \leq C_2 A(y)$ . If  $A \leq C_2 B$  or  $A \geq C_2 B$ , then we write  $A << B$  or  $A >> B$ .

## 2. AUXILIARY RESULTS

To prove the main results the following auxiliary propositions are used.

**Theorem B ([21])**. *Let  $p \in (1, \infty)$ . Then there exist positive numbers  $C_1(p), C_2(p)$  such that for any function  $f \in L_p(\mathbb{T}^m)$  the following inequality holds:*

$$\|f\|_p << \left\| \left( \sum_{s=0}^{\infty} |\delta_s(f)|^2 \right)^{\frac{1}{2}} \right\|_p << \|f\|_p.$$

**Theorem C ([22])**. *Let  $\bar{n} = (n_1, \dots, n_m)$ ,  $n_j \in \mathbb{N}, j = 1, \dots, m$  and*

$$T_{\bar{n}}(\bar{x}) = \sum_{|k_j| \leq n_j, j=1, \dots, m} c_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

*Then for  $1 \leq p_j < q_j < \infty, 1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty, j = 1, \dots, m$  the following inequality holds*

$$\|T_{\bar{n}}\|_{\bar{q}, \bar{\theta}^{(2)}} << \prod_{j=1}^m n_j^{\frac{1}{p_j} - \frac{1}{q_j}} \|T_{\bar{n}}\|_{\bar{p}, \bar{\theta}^{(1)}}.$$

Let  $\Omega_M$  be a set containing no more than  $M$  vectors  $\bar{k} = (k_1, \dots, k_m)$  with integer coordinates, and  $P(\Omega_M, \bar{x})$  be any trigonometric polynomial, which consists of harmonics with “indices” in  $\Omega_M$ .

**Lemma 1** (see [18]). *Let  $2 < q_j < +\infty, j = 1, \dots, m$ . Then for any trigonometric polynomial  $P(\Omega_N)$  and for any natural number  $M < N$  there exists trigonometric polynomial  $P(\Omega_M)$ , such that the following estimation holds*

$$\|P(\Omega_N) - P(\Omega_M)\|_{\bar{q}} << (NM^{-1})^{\frac{1}{2}} \|P(\Omega_N)\|_2,$$

and, moreover,  $\Omega_M \subset \Omega_N$ .

## 3. MAIN RESULTS

Let us prove the main results.

**Theorem 1.** . *Let  $\bar{p} = (p_1, \dots, p_m), \bar{q} = (q_1, \dots, q_m), \bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)}), \bar{\theta}^{(2)} = (\theta_1^{(2)}, \dots, \theta_m^{(2)}), 1 < p_j \leq 2 < q_j < \infty, 1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty, j = 1, \dots, m, 1 \leq \tau \leq \infty$ .*

1. *If  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^m \frac{1}{p_j}$ , then*

$$e_M \left( B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r \right)_{\bar{q}, \bar{\theta}^{(2)}} \asymp M^{-\left(2 \sum_{j=1}^m \frac{1}{q_j} - 1\right) \left(r - \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)\right)}.$$

2. *If  $r = \sum_{j=1}^m \frac{1}{p_j}$ , then*

$$e_M \left( B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r \right)_{\bar{q}, \bar{\theta}^{(2)}} \asymp M^{-\frac{1}{2}} (\log_2 M)^{1 - \frac{1}{\tau}},$$

for  $M > 1$ .

3. *If  $r > \sum_{j=1}^m \frac{1}{p_j}$ , then*

$$e_M \left( B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r \right)_{\bar{q}, \bar{\theta}^{(2)}} \asymp M^{-\frac{1}{m} \left(r + \sum_{j=1}^m \left(\frac{1}{2} - \frac{1}{p_j}\right)\right)}.$$

**Proof.** Firstly, we are going to consider the upper bound in the first item. Taking into account the inclusion  $B_{\bar{p}, \bar{\theta}(1), \tau}^r \subset H_{\bar{p}, \bar{\theta}(1)}^r$ ,  $1 \leq \tau < +\infty$ , it suffices to prove it for the class  $H_{\bar{p}, \bar{\theta}(1)}^r$ .

Let  $1 < p_j \leq 2 < q_j < \infty$ ,  $j = 1, \dots, m$ , and  $\mathbb{N}$  be the set of natural numbers. For a number  $M \in \mathbb{N}$  choose a natural number  $n$  such that  $2^{nm} < M \leq 2^{(n+1)m}$ . For a function  $f \in H_{\bar{p}, \bar{\theta}(1)}^r$ , it is known that

$$\|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)} \leq 2^{-sr}, \quad 1 < p_j < \infty, \quad j = 1, \dots, m.$$

We will seek an approximation polynomial  $P(\Omega_M, \bar{x})$  in the form

$$P(\Omega_M, \bar{x}) = \sum_{s=0}^{n-1} \delta_s(f, \bar{x}) + \sum_{n \leq s < \alpha n} P(\Omega_{N_s}, \bar{x}), \quad (1)$$

where the polynomials  $P(\Omega_{N_s}, \bar{x})$  will be constructed for each  $\delta_s(f, \bar{x})$  in accordance with Lemma 1, and the number  $\alpha > 1$  will be chosen during the construction.

Let  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^m \frac{1}{p_j}$ . Suppose

$$N_s = \left[ 2^{nm} 2^{s(\sum_{j=1}^m \frac{1}{p_j} - r)} 2^{-n\alpha(\sum_{j=1}^m \frac{1}{p_j} - r)} \right] + 1,$$

where  $[y]$  integer part of the number  $y$ .

Now we are going to show that polynomials (1) have no more than  $M$  harmonics (in terms of order). By definition of the number  $N_s$ , we have

$$\begin{aligned} & \sum_{s=0}^{n-1} \#\{\bar{k} = (k_1, \dots, k_m) : [2^{s-1}] \leq \max_{j=1, \dots, m} |k_j| < 2^s\} + \sum_{n \leq s < \alpha n} N_s \leq C 2^{nm} + \\ & + \sum_{n \leq s < \alpha n} \left( 2^{nm} 2^{s(\sum_{j=1}^m \frac{1}{p_j} - r)} 2^{-n\alpha(\sum_{j=1}^m \frac{1}{p_j} - r)} + 1 \right) < 2^{nm} + (\alpha - 1)n < 2^{nm} \asymp M, \end{aligned}$$

where  $\#A$  denotes the number of elements of the set  $A$ .

Next, by the property of the norm we have

$$\begin{aligned} \|f - P(\Omega_M)\|_{\bar{q}, \bar{\theta}(2)} & \leq \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\bar{q}, \bar{\theta}(2)} + \\ & + \left\| \sum_{\alpha n \leq s < +\infty} \delta_s(f) \right\|_{\bar{q}, \bar{\theta}(2)} = J_1(n) + J_2(n). \end{aligned} \quad (2)$$

Let us estimate  $J_2(n)$ . Applying the inequality of different metrics for trigonometric polynomials (see Theorem C), we can obtain

$$J_2(n) \leq \sum_{\alpha n \leq s < +\infty} \|\delta_s(f)\|_{\bar{q}, \bar{\theta}(2)} < \sum_{\alpha n \leq s < +\infty} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)}.$$

Therefore, taking into account  $f \in H_{\bar{p}, \bar{\theta}(1)}^r$  and  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r$ , we get

$$J_2(n) < \sum_{\alpha n \leq s < +\infty} 2^{-s(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} < 2^{-n\alpha(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}. \quad (3)$$

Let us estimate  $J_1(n)$ . Using the property of the quasi-norm, Lemma 1 and the inequality of different metrics (see Theorem C), we get

$$\begin{aligned}
J_1(n) &= \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\bar{q}, \bar{\theta}(2)} << \sum_{n \leq s < \alpha n} \|\delta_s(f) - P(\Omega_{N_s})\|_{\bar{q}, \bar{\theta}(2)} << \\
&<< \sum_{n \leq s < \alpha n} (N_s^{-1} 2^{sm})^{\frac{1}{2}} \|\delta_s(f)\|_2 << \\
&<< \sum_{n \leq s < \alpha n} (N_s^{-1} 2^{sm})^{\frac{1}{2}} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)} << \\
&<< \sum_{n \leq s < \alpha n} N_s^{-\frac{1}{2}} 2^{s \sum_{j=1}^m \frac{1}{p_j}} 2^{-sr} << \\
&<< 2^{-\frac{nm}{2}} 2^{\frac{n\alpha}{2} (\sum_{j=1}^m \frac{1}{p_j} - r)} \sum_{n \leq s < \alpha n} 2^{s (\sum_{j=1}^m \frac{1}{p_j} - r) \frac{1}{2}} << 2^{-\frac{nm}{2}} 2^{n\alpha (\sum_{j=1}^m \frac{1}{p_j} - r)}. \tag{4}
\end{aligned}$$

Suppose  $\alpha = m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}$ . Then from the inequality (4), we get

$$J_1(n) \leq C 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \asymp M^{-(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}. \tag{5}$$

For  $\alpha = m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}$ , using the inequality (3) and taking into account  $2^{nm} \asymp M$  we obtain

$$J_2(n) << M^{-(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}. \tag{6}$$

By (5) and (6), we get from the inequality (2) the following

$$\|f - P(\Omega_M)\|_{\bar{q}, \bar{\theta}(2)} << M^{-(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

for any function  $f \in H_{\bar{p}, \bar{\theta}(1)}^r$  in the case of  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^m \frac{1}{p_j}$ .

From the inclusion  $B_{\bar{p}, \bar{\theta}(1), \tau}^r \subset H_{\bar{p}, \bar{\theta}(1)}^r$  and the definition of  $M$  - term approximation, it follows that

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} << M^{-(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

in the case of  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^m \frac{1}{p_j}$ .

Let us consider the lower bound. We will use the well-known formula ([23], p.25)

$$e_M(f)_{\bar{q}, \bar{\theta}(2)} = \inf_{\Omega_M} \sup_{P \in L^\perp, \|P\|_{\bar{q}', \bar{\theta}(2)'} \leq 1} \left| \int_{\mathbb{T}^m} f(\bar{x}) P(\bar{x}) d\bar{x} \right|, \tag{7}$$

where  $\bar{q}' = (q'_1, \dots, q'_m)$ ,  $\bar{\theta}(2)' = (\theta_1^{(2)'}, \dots, \theta_m^{(2)'})$ ,  $\frac{1}{q_j} + \frac{1}{q_j'} = 1$ ,  $\frac{1}{\theta_j^{(2)}} + \frac{1}{\theta_j^{(2)'}} = 1$ ,  $j = 1, \dots, m$ , and  $L_M^\perp$  is the set of functions that are orthogonal to the subspace of trigonometric polynomials with harmonics in the set  $\Omega_M$ .

Consider the function

$$F_{\bar{q},n}(\bar{x}) = \sum_{\substack{\max_{j=1,\dots,m} |k_j| \leq 2 \\ [nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}]}} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Let  $\Omega_M$  be a set of  $M$  vectors with integer coordinates. Suppose

$$g(\bar{x}) = F_{\bar{q},n}(\bar{x}) - \sum_{\bar{k} \in \Omega_M}^* e^{i\langle \bar{k}, \bar{x} \rangle},$$

where the sum  $\sum_{\bar{k} \in \Omega_M}^*$  contains those terms in the function  $F_{\bar{q},n}(\bar{x})$  with indices only in  $\Omega_M$ . By the inequality

$$\left\| \sum_{\substack{\max_{j=1,\dots,m} |k_j| \leq 2^l}} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}(1)} << 2^{l \sum_{j=1}^m (1 - \frac{1}{p_j})} \quad (8)$$

and the Parseval's equality for  $1 < q_j' < 2, j = 1, \dots, m$ , we obtain

$$\begin{aligned} \|g\|_{\bar{q}', \bar{\theta}(2)'} &\leq \|F_{\bar{q},n}\|_{\bar{q}', \bar{\theta}(2)'} + (2\pi)^{\sum_{j=1}^m (\frac{1}{q_j} - \frac{1}{2})} \left\| \sum_{\bar{k} \in \Omega_M}^* e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_2 << \\ &<< (2^{\frac{nm}{2}} + M^{\frac{1}{2}}) << 2^{\frac{nm}{2}}. \end{aligned} \quad (9)$$

Now we consider the function

$$P_1(\bar{x}) = C_2 2^{-\frac{nm}{2}} g(\bar{x}). \quad (10)$$

Then (9) implies follows that the function  $P_1$  satisfies the assumptions of the formula (7) for some constant  $C_2 > 0$ .

Consider the function

$$f_n(\bar{x}) = C_3 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} F_{\bar{q},n}(\bar{x}).$$

By the inequality (8), we get

$$\begin{aligned} &\sum_{s=0}^{\infty} 2^{sr} \|\delta_s(f_n)\|_{\bar{p}, \bar{\theta}(1)} << \\ &<< 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} \sum_{s=0}^{[m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}]} 2^{sr} \|\delta_s(F_{\bar{q},n})\|_{\bar{p}, \bar{\theta}(1)} << \\ &<< 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} \sum_{s=0}^{[m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}]} 2^{sr} 2^{\sum_{j=1}^m (1 - \frac{1}{p_j})} \leq C_3. \end{aligned}$$

Hence, the function  $C_3^{-1} f_n \in B_{\bar{p}, \bar{\theta}(1), 1}^r$ .

For the functions (10) and (11), we have by the formula (7), the following

$$\begin{aligned} e_M(f_n)_{\bar{q}, \bar{\theta}(2)} &>> \inf_{\Omega_M} \left| \int_{\mathbb{T}^m} f_n(\bar{x}) P_1(\bar{x}) d\bar{x} \right| >> \\ &>> 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} 2^{-\frac{nm}{2}} (\|F_{\bar{q},n}\|_2^2 - M) >> \end{aligned}$$

$$\begin{aligned}
&>> 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} 2^{-\frac{nm}{2}} 2^{nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}} = \\
&= C 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}.
\end{aligned} \tag{12}$$

Hence, it follows from (12) by the inclusion  $B_{\vec{p}, \vec{\theta}(1), 1}^r \subset B_{\vec{p}, \vec{\theta}(1), \tau}^r$  that

$$e_M(f_n)_{\vec{q}, \vec{\theta}(2)} >> 2^{-nm(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

in the case of  $\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}) < r < \sum_{j=1}^m \frac{1}{p_j}$ .

So we have proved the first item.

Now we consider the case  $r = \sum_{j=1}^m \frac{1}{p_j}$ . Let  $f \in B_{\vec{p}, \vec{\theta}(1), \tau}^r$ . Suppose

$$\alpha = m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1} \text{ and}$$

$$N_s = \left[ 2^{nm} n^{\frac{1}{\tau}-1} \|\delta_s(f_n)\|_{\vec{p}, \vec{\theta}(1)} 2^{sr} \right] + 1.$$

Then, by definition of the numbers  $N_s$  and by Holders inequality, we obtain

$$\begin{aligned}
&\sum_{s=0}^{n-1} \# \rho(s) + \sum_{n \leq s < \alpha n} N_s << \\
&<< 2^{nm} + (\alpha - 1)n + 2^{nm} n^{\frac{1}{\tau}-1} \sum_{n \leq s < \alpha n} \|\delta_s(f_n)\|_{\vec{p}, \vec{\theta}(1)} 2^{sr} << \\
&<< 2^{nm} + (\alpha - 1)n + 2^{nm} n^{\frac{1}{\tau}-1} ((\alpha - 1)n)^{\frac{1}{\tau'}} \left( \sum_{s=0}^{\infty} \|\delta_s(f_n)\|_{\vec{p}, \vec{\theta}(1)}^{\tau} 2^{sr\tau} \right)^{\frac{1}{\tau}} << \\
&<< 2^{nm} \asymp M.
\end{aligned}$$

To estimate  $J_1(n)$  let  $\beta = \max\{q_1, \dots, q_m\}$ . Then  $\beta > 2$  and  $L_\beta(\mathbb{T}^m) \subset L_{\vec{q}, \vec{\theta}(2)}(\mathbb{T}^m)$ . Therefore by applying Theorem B and by the norm property we obtain

$$\begin{aligned}
J_1(n) &= \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\vec{q}, \vec{\theta}(2)} \leq C \left\| \sum_{n \leq s < \alpha n} (\delta_s(f) - P(\Omega_{N_s})) \right\|_{\beta} << \\
&<< \left\| \left( \sum_{n \leq s < \alpha n} |\delta_s(f) - P(\Omega_{N_s})|^2 \right)^{\frac{1}{2}} \right\|_{\beta} << \\
&<< \left( \sum_{n \leq s < \alpha n} \|\delta_s(f) - P(\Omega_{N_s})\|_{\beta}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

It implies by Lemma 1 and by the inequality of different metrics (see Theorem C) that

$$\begin{aligned}
J_1(n) &<< \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{sm} \|\delta_s(f)\|_2^2 \right)^{\frac{1}{2}} << \\
&<< \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{sm} 2^{2s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})} \|\delta_s(f)\|_{\vec{p}, \vec{\theta}(1)}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Next, since  $r = \sum_{j=1}^m \frac{1}{p_j}$ , we have by definition of the numbers  $N_s$  and using Holders inequality, the following

$$\begin{aligned} J_1(n) &<< \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{sm} 2^{2s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)}^2 \right)^{\frac{1}{2}} << \\ &<< (2^{-nm} n^{1-\frac{1}{\theta}})^{\frac{1}{2}} \left( \sum_{n \leq s < \alpha n} 2^{sr} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)} \right)^{\frac{1}{2}} << \\ &<< (2^{-nm} n^{1-\frac{1}{\theta}})^{\frac{1}{2}} \left( \sum_{n \leq s < \alpha n} 2^{sr\theta} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)}^{\tau} \right)^{\frac{1}{2\tau}} (\alpha - 1)^{\frac{1}{2\tau}} = \\ &= C 2^{-\frac{nm}{2}} n^{1-\frac{1}{\tau}} \asymp M^{-\frac{1}{2}} (\log M)^{1-\frac{1}{\tau}}. \end{aligned}$$

Thus

$$J_1(n) << M^{-\frac{1}{2}} (\log M)^{1-\frac{1}{\tau}} \quad (13)$$

in the case of  $r = \sum_{j=1}^m \frac{1}{p_j}$ .

For the estimation of  $J_2(n)$  we apply Holders inequality and taking into account that  $r = \sum_{j=1}^m \frac{1}{p_j}$  and  $\alpha = m(2 \sum_{j=1}^m \frac{1}{q_j})^{-1}$ , we obtain

$$\begin{aligned} J_2(n) &<< \sum_{n\alpha \leq s < +\infty} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)} << \\ &<< \left( \sum_{s=0}^{\infty} 2^{sr\tau} \|\delta_s(f)\|_{\bar{p}, \bar{\theta}(1)}^{\tau} \right)^{\frac{1}{\theta}} \left( \sum_{n\alpha \leq s < +\infty} 2^{-s\tau' \sum_{j=1}^m \frac{1}{q_j}} \right)^{\frac{1}{\tau'}} << \\ &<< 2^{-n\alpha \sum_{j=1}^m \frac{1}{q_j}} = C 2^{-\frac{nm}{2}} \asymp M^{-\frac{1}{2}}, \end{aligned} \quad (14)$$

where  $\tau' = \frac{\tau}{\tau-1}$ .

By (13) and (14) the inequality (2) implies that

$$\|f - P(\Omega_M)\|_{\bar{q}, \bar{\theta}(2)} << M^{-\frac{1}{2}} (\log M)^{1-\frac{1}{\tau}}$$

in the case  $r = \sum_{j=1}^m \frac{1}{p_j}$ . It proves the upper bound estimation in the second item.

Let  $r > \sum_{j=1}^m \frac{1}{p_j}$ . Suppose

$$N_s = \left[ 2^{n(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} 2^{-s(r - \sum_{j=1}^m \frac{1}{p_j})} \right] + 1.$$

Then

$$\begin{aligned} &\sum_{s=0}^{n-1} \sharp \rho(s) + \sum_{n \leq s < \alpha n} N_s << \\ &<< 2^{nm} + (\alpha - 1)n + 2^{n(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} \sum_{n \leq s < \alpha n} 2^{-s(r - \sum_{j=1}^m \frac{1}{p_j})} << \end{aligned}$$



$$<< 2^{nm} + (\alpha - 1)n << 2^{nm} << M.$$

If  $f \in H_{\vec{p}, \vec{\theta}(1)}^r$ , then, by definition of the numbers  $N_s$  and  $r > \sum_{j=1}^m \frac{1}{p_j}$  we obtain

$$\begin{aligned} J_1(n) &\leq \left( \sum_{n \leq s < \alpha n} N_s^{-1} 2^{sm} 2^{2s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})} \|\delta_s(f)\|_{\vec{p}, \vec{\theta}(1)}^2 \right)^{\frac{1}{2}} << \\ &<< 2^{-\frac{n}{2}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} \left( \sum_{n \leq s < \alpha n} 2^{s(r + \sum_{j=1}^m \frac{1}{p_j})} \|\delta_s(f)\|_{\vec{p}, \vec{\theta}(1)}^2 \right)^{\frac{1}{2}} << \\ &<< 2^{-\frac{n}{2}(r - \sum_{j=1}^m (\frac{1}{p_j} - 1))} \left( \sum_{n \leq s < \alpha n} 2^{-s(r - \sum_{j=1}^m \frac{1}{p_j})} \right)^{\frac{1}{2}} << 2^{-n(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}. \end{aligned}$$

Thus,

$$J_1(n) << M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))} \quad (15)$$

in the case of  $r > \sum_{j=1}^m \frac{1}{p_j}$ .

To estimate  $J_2(n)$ , we suppose  $\alpha = (r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))(r + \sum_{j=1}^m (\frac{1}{q_j} - \frac{1}{p_j}))^{-1}$  and get

$$\begin{aligned} J_2(n) &<< \sum_{n\alpha \leq s < \infty} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})} \|\delta_s(f)\|_{\vec{p}, \vec{\theta}(1)} \leq \sum_{n\alpha \leq s < \infty} 2^{-s(r + \sum_{j=1}^m (\frac{1}{q_j} - \frac{1}{p_j}))} << \\ &<< 2^{-n\alpha(r + \sum_{j=1}^m (\frac{1}{q_j} - \frac{1}{p_j}))} << 2^{-n(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))} << M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))} \end{aligned} \quad (16)$$

for a function  $f \in H_{\vec{p}, \vec{\theta}(1)}^r$ . By (15) and (14), it follows from (2) that

$$\|f - P(\Omega_M)\|_{\vec{q}, \vec{\theta}(2)} << M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}$$

for any function  $f \in H_{\vec{p}, \vec{\theta}(1)}^r$  in the case of  $r > \sum_{j=1}^m \frac{1}{p_j}$ .

From  $B_{\vec{p}, \vec{\theta}(1), \tau}^r \subset H_{\vec{p}, \vec{\theta}(1)}^r$  it follows that

$$e_M \left( B_{\vec{p}, \vec{\theta}(1), \tau}^r \right)_{\vec{q}, \vec{\theta}(2)} << M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}$$

in the case of  $r > \sum_{j=1}^m \frac{1}{p_j}$ . It proves the upper bound estimation in the item 3.

Let us consider the lower bound estimation in the case  $r = \sum_{j=1}^m \frac{1}{p_j}$ . Consider the function

$$g_1(\vec{x}) = \sum_{s=1}^n \sum_{\vec{k} \in \rho(s)} \prod_{j=1}^m k_j^{-1} \cos k_j x_j. \quad (17)$$

Then

$$\delta_s(g_1, \vec{x}) = \sum_{\vec{k} \in \rho(s)} \prod_{j=1}^m k_j^{-1} \cos k_j x_j.$$

It is known that for a function  $d_s(\bar{x}) = \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m \cos k_j x_j$  the following relation holds

$$\|d_s\|_{\bar{p}, \bar{\theta}^{(1)}} \asymp 2^{s \sum_{j=1}^m (1 - \frac{1}{p_j})}, \quad 1 < p_j, \theta_j^{(1)} < +\infty, \quad j = 1, \dots, m.$$

Therefore by the inequality of distinct metrics (see Theorem C) and by the Marcinkiewicz theorem on multipliers, we have

$$\|\delta_s(g_1)\|_{\bar{p}, \bar{\theta}^{(1)}} << 2^{-sm} \|d_s\|_{\bar{p}, \bar{\theta}^{(1)}} \leq C 2^{-s \sum_{j=1}^m \frac{1}{p_j}}.$$

Hence, since  $r = \sum_{j=1}^m \frac{1}{p_j}$  we obtain

$$\left( \sum_{s=0}^{\infty} 2^{sr\tau} \|\delta_s(g_1)\|_{\bar{p}, \bar{\theta}^{(1)}}^{\tau} \right)^{\frac{1}{\tau}} \leq C_1 n^{\frac{1}{\tau}}.$$

Therefore the function  $f_1(\bar{x}) = C_1^{-1} n^{-\frac{1}{\tau}} g_1(\bar{x})$  belongs to the class  $B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r$ ,  $1 < p_j < +\infty, j = 1, \dots, m$ .

Now, we are going to construct a function  $P_1$ , which satisfies the conditions of the formula (7). Let

$$v_1(\bar{x}) = \sum_{s=1}^n \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m \cos k_j x_j$$

and  $\Omega_M$  be an arbitrary set of vectors  $\bar{k} = (k_1, \dots, k_m)$  in  $M$  with integer coordinates. Consider the function

$$u_1(\bar{x}) = \sum_{s=1}^n \sum_{\bar{k} \in \rho(s) \cap \Omega_M} \prod_{j=1}^m \cos k_j x_j.$$

Suppose  $w_1(\bar{x}) = v_1(\bar{x}) - u_1(\bar{x})$ . Then, since  $1 < q_j' = \frac{q_j}{q_j-1} < 2$ ,  $j = 1, \dots, m$ , we obtain, by the Perseval's equality, the following

$$\|w_1\|_{\bar{q}', \bar{\theta}^{(2)'}} \leq \|v_1\|_{\bar{q}', \bar{\theta}^{(2)'}} + \|u_1\|_2 \leq \|v_1\|_{\bar{q}', \bar{\theta}^{(2)'}} + CM^{\frac{1}{2}}.$$

By the property of quasi-norm and the estimation of the norm of the Dirichlet kernel in the Lorentz space, we have

$$\begin{aligned} \|v_1\|_{\bar{q}', \bar{\theta}^{(2)'}} &<< \sum_{s=1}^n \|\delta_s(g_1)\|_{\bar{q}', \bar{\theta}^{(2)'}} << \\ &<< \sum_{s=1}^n 2^{s \sum_{j=1}^m (1 - \frac{1}{q_j'})} << 2^{n \sum_{j=1}^m (1 - \frac{1}{q_j'})} = C 2^{n \sum_{j=1}^m \frac{1}{q_j'}}. \end{aligned}$$

Therefore, taking into account  $\frac{1}{q_j} < \frac{1}{2}$ ,  $j = 1, \dots, m$ , we get

$$\|w_1\|_{\bar{q}', \bar{\theta}^{(2)'}} << (2^{\frac{nm}{2}} + M^{\frac{1}{2}}) \leq C 2^{\frac{nm}{2}}.$$

Hence the function

$$P_1(\bar{x}) = C_2^{-1} 2^{-\frac{nm}{2}} w_1(\bar{x})$$

satisfies the conditions of the formula (7). Then, by substituting the functions  $f_1$  and  $P_1$  into (7) and by orthogonality of a trigonometric system, we obtain

$$\begin{aligned} e_M(f_1)_{\bar{q}, \bar{\theta}(2)} &>> \sum_{n_1 \leq s < n} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m k_j^{-1} 2^{-\frac{nm}{2}} n^{-\frac{1}{\tau}} >> \\ &\geq C(\ln 2)^m \sum_{n_1 \leq s < n} 2^{-\frac{nm}{2}} n^{-\frac{1}{\tau}} = C(\ln 2)^m 2^{-\frac{nm}{2}} n^{-\frac{1}{\tau}} (n - n_1) \geq \\ &>> (\ln 2)^m 2^{-\frac{nm}{2}} n^{1-\frac{1}{\tau}} \asymp M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{\tau}}, \end{aligned}$$

where  $n_1$  is a natural number such that  $n_1 < \frac{n}{2}$ .

So, for the function  $f_1 \in B_{\bar{p}, \bar{\theta}(1), \tau}^r$  it has been proved that

$$e_M(f_1)_{\bar{q}, \bar{\theta}(2)} >> M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{\tau}}$$

in the case of  $r = \sum_{j=1}^m \frac{1}{p_j}$ . Hence

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} >> M^{-\frac{1}{2}} (\log_2 M)^{1-\frac{1}{\tau}}$$

in the case of  $r = \sum_{j=1}^m \frac{1}{p_j}$ . It proves the lower bound estimation in the second item.

Let us prove the lower bound estimation for the case  $r > \sum_{j=1}^m \frac{1}{p_j}$ . Since in this case an upper bound estimation of the quantity  $e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)}$  does not depend on  $\tau$  and  $B_{\bar{p}, \bar{\theta}(1), 1}^r \subset B_{\bar{p}, \bar{\theta}(1), \tau}^r$ ,  $1 < \tau < +\infty$ , it suffices to prove the lower bound estimation for  $B_{\bar{p}, \bar{\theta}(1), 1}^r$ .

For a number  $M \in \mathbb{N}$ , we choose a natural number  $n$  such that  $2^{nm} < M \leq 2^{(n+1)m}$  and  $2M \leq \sharp \rho(n)$ , where  $\sharp \rho(n)$  denotes the number of elements in the set  $\rho(n)$ .

Consider the following function

$$f_3(\bar{x}) = n^{-1} \sum_{s=1}^n 2^{-s \sum_{j=1}^m (1-\frac{1}{p_j})} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m k_j^{-\frac{r}{m}} \cos k_j x_j.$$

Then

$$\|\delta_s(f_3)\|_{\bar{p}, \bar{\theta}(1)} << 2^{-sr} n^{-1}.$$

Hence

$$\sum_{s=0}^{\infty} 2^{sr} \|\delta_s(f_3)\|_{\bar{p}, \bar{\theta}(1)} \leq C_3$$

i.e. the function  $C_3^{-1} f_3 \in B_{\bar{p}, \bar{\theta}(1), 1}^r$ .

Next, consider the functions

$$\begin{aligned} v_3(\bar{x}) &= \sum_{s=1}^n \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m \cos k_j x_j, \\ u_3(\bar{x}) &= \sum_{s=1}^n \sum_{\bar{k} \in \rho(s) \cap \Omega_M} \prod_{j=1}^m \cos k_j x_j. \end{aligned}$$

Suppose  $w_3(\bar{x}) = v_3(\bar{x}) - u_3(\bar{x})$ . By the Perseval's equality,

$$\begin{aligned}\|u_3\|_2 &\leq M^{\frac{1}{2}}, \\ \|v_3\|_2 &= 2^{\frac{(n-1)m}{2}}.\end{aligned}$$

From these relations, we obtain, by the properties of the norm, the following

$$\|w_3\|_2 \leq \|v_3\|_2 + \|u_3\|_2 \leq C_4 2^{\frac{nm}{2}}.$$

Therefore the function  $P_3(\bar{x}) = C_4^{-1} 2^{-\frac{nm}{2}} w_3(\bar{x})$  satisfies the conditions of the formula (7). Since  $2 < q_j$   $j = 1, \dots, m$ , we have  $e_M(f_3)_2 \leq C e_M(f_3)_{\bar{q}, \bar{\theta}(2)}$ . Now, by the formula (7), we get

$$\begin{aligned}e_M(f_3)_{\bar{q}, \bar{\theta}(2)} &>> e_M(f_3)_2 >> n^{-1} 2^{-\frac{nm}{2}} \sum_{s=1}^n 2^{-s \sum_{j=1}^m (1 - \frac{1}{p_j})} \sum_{\bar{k} \in \rho(s)} \prod_{j=1}^m k_j^{-\frac{r}{m}} >> \\ &>> n^{-1} 2^{-\frac{nm}{2}} \sum_{s=1}^n 2^{-s \sum_{j=1}^m (1 - \frac{1}{p_j})} 2^{s(m-r)} = \\ &= C n^{-1} 2^{-\frac{nm}{2}} \sum_{s=1}^n 2^{-s(r - \sum_{j=1}^m \frac{1}{p_j})} >> 2^{-n(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}.\end{aligned}$$

It follows from the relation  $2^{nm} \asymp M$  that

$$e_M(f_3)_{\bar{q}, \bar{\theta}(2)} >> M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}$$

in the case  $r > \sum_{j=1}^m \frac{1}{p_j}$  for the function  $C_3^{-1} f_3 \in B_{\bar{p}, \bar{\theta}^{(1)}, 1}^r$ . Hence

$$e_M\left(B_{\bar{p}, \bar{\theta}^{(1)}, 1}^r\right)_{\bar{q}, \bar{\theta}(2)} >> M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}.$$

Therefore

$$e_M\left(B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r\right)_{\bar{q}} >> M^{-\frac{1}{m}(r + \sum_{j=1}^m (\frac{1}{2} - \frac{1}{p_j}))}$$

in the case  $r > \sum_{j=1}^m \frac{1}{p_j}$ . So Theorem 1 has been proved.

**Theorem 2.** . Let  $\bar{p} = (p_1, \dots, p_m)$ ,  $\bar{q} = (q_1, \dots, q_m)$ ,  $1 < p_j < q_j \leq 2$ ,  $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$ ,  $j = 1, \dots, m$ ,  $1 \leq \tau \leq +\infty$ .

If  $r > \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})$ , then

$$e_M\left(B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r\right)_{\bar{q}, \theta^{(2)}} \asymp M^{-\frac{1}{m}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}.$$

**Proof.** For a number  $M \in \mathbb{N}$  choose a natural number  $n$  such that  $M \asymp 2^{nm}$ . By the inequality of distinct metrics and by Holder's inequality, we have

$$\begin{aligned}\|f - \sum_{s=0}^n \delta_s(f)\|_{\bar{q}, \bar{\theta}(2)} &\leq \sum_{s=n}^{\infty} \|\delta_s(f)\|_{\bar{q}, \bar{\theta}(2)} \leq \\ &\leq \left[ \sum_{s=0}^{\infty} 2^{s\tau r} \|\delta_s(f)\|_{\bar{q}, \bar{\theta}(2)}^{\tau} \right]^{\frac{1}{\tau}} \leq \left[ \sum_{s=n}^{\infty} 2^{s\tau'(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \right]^{\frac{1}{\tau}} <<\end{aligned}$$

$$<< 2^{n(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))} \leq CM^{-\frac{1}{m}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}$$

for  $f \in B_{\bar{p}, \bar{\theta}(1), \tau}^r$ ,  $\frac{1}{\tau} + \frac{1}{\tau} = 1$ . Therefore

$$e_M(f)_{\bar{q}, \bar{\theta}(2)} \leq \|f - \sum_{s=0}^n \delta_s(f)\|_{\bar{q}, \bar{\theta}(2)} << M^{-\frac{1}{m}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}.$$

Hence

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} << M^{-\frac{1}{m}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}.$$

It proves the upper bound estimation.

For the lower bound estimation, let us consider the function

$$f_0(\bar{x}) = n^{-r + \sum_{j=1}^m (\frac{1}{p_j} - 1)} V_n(\bar{x}),$$

where  $V_n(\bar{x})$  is a Valle-Poisson sum with multiplicity.

Next, following the proof in [19] (pp. 46-47) and applying Theorem B, we obtain the lower bound estimation of the quantity  $e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)}$ .

**Theorem 3.** . Let  $\bar{p} = (p_1, \dots, p_m)$ ,  $\bar{q} = (q_1, \dots, q_m)$ ,  $2 \leq p_j < q_j < \infty$ ,  $1 < \theta_j^{(1)}, \theta_j^{(2)} < \infty$ ,  $j = 1, \dots, m$ ,  $1 \leq \tau \leq +\infty$ . If  $r > \frac{m}{2}$ , then

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} \asymp M^{-\frac{r}{m}}.$$

**Proof.** By the inclusion  $B_{\bar{p}, \bar{\theta}(1), \tau}^r \subset B_{2, \bar{\theta}(1), \tau}^r \subset H_{2, \bar{\theta}(1)}^r$ , we have

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} \leq e_M \left( B_{2, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} \leq e_M \left( H_{2, \bar{\theta}(1)}^r \right)_{\bar{q}, \bar{\theta}(2)}.$$

By Theorem 1,

$$e_M \left( H_{2, \bar{\theta}(1)}^r \right)_{\bar{q}, \bar{\theta}(2)} << M^{-\frac{r}{m}}.$$

for  $p_j = 2, j = 1, \dots, m$ . Hence

$$e_M \left( B_{\bar{p}, \bar{\theta}(1), \tau}^r \right)_{\bar{q}, \bar{\theta}(2)} << M^{-\frac{r}{m}}.$$

it proves the upper bound estimation.

Let us consider the lower bound estimation. Consider Rudin-Shapiros polynomial (see [24], p.155) of the type

$$R_s(x) = \sum_{k=2^{s-1}}^{2^s} \varepsilon_k e^{ikx}, \quad x \in [0, 2\pi], \quad \varepsilon_k = \pm 1.$$

it is known that  $\|R_s\|_\infty = \max_{x \in [0, 2\pi]} |R_s(x)| << 2^{\frac{s}{2}}$  (see [24], p. 155). For a given number  $M$  choose a number  $n$  such that  $M \asymp 2^{nm}$ . Now consider the function

$$f_0(\bar{x}) = 2^{-n(\frac{m}{2} + r)} \sum_{s=1}^n \prod_{j=1}^m R_s(x_j)$$

Then, by the continuity,  $f_0 \in L_{\bar{p}, \bar{\theta}(1)}(\mathbb{T}^m)$  and

$$\sum_{s=0}^\infty 2^{s\tau r} \|\delta_s(f_0)\|_{\bar{p}, \bar{\theta}(1)}^\tau = 2^{-n(\frac{m}{2} + r)} \sum_{s=1}^n 2^{s\tau r} \left\| \prod_{j=1}^m R_s(x_j) \right\|_{\bar{p}, \bar{\theta}(1)}^\tau \leq$$

$$\leq 2^{-n(\frac{m}{2}+r)} \sum_{s=1}^n 2^{s(\frac{m}{2}+r)\tau} \leq C_0.$$

Hence, the function  $C_0^{-1}f_0 \in B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r$ . Now construct a function  $P(\bar{x})$ , which would satisfy the conditions in the formula (7). Suppose

$$v_0(\bar{x}) = \sum_{s=1}^n \prod_{j=1}^m R_s(x_j), \quad u_0(\bar{x}) = \sum_{s=1}^* \prod_{j=1}^m R_s(x_j),$$

where the sign  $*$  means that the polynomial  $u_0(\bar{x})$  contains only those harmonics of  $v_0$  which have indices in  $\Omega_M$ . Suppose  $w_0(\bar{x}) = v_0(\bar{x}) - u_0(\bar{x})$ . Then, since  $1 < q'_j = \frac{q_j}{q_j-1} < 2$ ,  $j = 1, \dots, m$ , and by the Percevals equality, we have

$$\|w_0\|_{\bar{q}', \bar{\theta}^{(2)'}} \leq \|w_0\|_2 \leq C_1 2^{\frac{nm}{2}}.$$

Therefore, for the function  $P_0(\bar{x}) = C_1^{-1} 2^{-\frac{nm}{2}} w_0(\bar{x})$ , the inequality holds  $\|P_0\|_{\bar{q}', \theta^{(2)'}} \leq 1$ . Now using the formula (7), we obtain

$$\begin{aligned} e_M \left( B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r \right)_{\bar{q}, \bar{\theta}^{(2)}} &>> e_M(f_0)_{\bar{q}, \bar{\theta}^{(2)}} >> 2^{-n(\frac{m}{2}+r)} 2^{-\frac{nm}{2}} (2^{nm} - M) >> \\ &>> 2^{-n(m+r)} 2^{nm} >> M^{-\frac{r}{m}}. \end{aligned}$$

So

$$e_M \left( B_{\bar{p}, \bar{\theta}^{(1)}, \tau}^r \right)_{\bar{q}, \bar{\theta}^{(2)}} >> M^{-\frac{r}{m}}.$$

It proves Theorem 3.

**Corollary.** Let  $1 < p \leq 2 < q < \infty$ ,  $1 \leq \tau \leq \infty$  and  $r = \frac{m}{p}$ . Then

$$e_M \left( B_{p, \tau}^r \right)_q \asymp M^{-\frac{1}{2}} (\log M)^{1-\frac{1}{\theta}}.$$

The proof follows from the second item of Theorem 2.1 if  $p_j = \theta_j^{(1)} = p$ ,  $q_j = \theta_j^{(2)} = q$ ,  $j = 1, \dots, m$ .

**Remark.** In the case  $p_j = \theta_j^{(1)} = p$ ,  $q_j = \theta_j^{(2)} = q$ ,  $j = 1, \dots, m$  and  $r > m(\frac{1}{p} - \frac{1}{q})$ , the results of R.A. DeVore and V.N. Temlyakov [20] follow from Theorem 1 - 3. If  $1 < p \leq 2 < q < \infty$  and  $m(\frac{1}{p} - \frac{1}{q}) < r \leq \frac{m}{p}$ , the results of S.A. Stasyuk [16], [17] follow from the first and second items of Theorem 1.

The cases  $p_j = \theta_j^{(1)}$ ,  $q_j = \theta_j^{(2)}$ ,  $j = 1, \dots, m$ . of Theorem 1 - 3 were announced in [25] and in of Theorem 1 the first item proved [26].

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DEPARTMENT OF MATHEMATICS AND INFORMATION TECHNOLOGY, BUKETOV KARAGANDA STATE UNIVERSITY, UNIVERSYTETSKAYA 28 , 100028, KARAGANDA , REPUBLIC KAZAKHSTAN